**3-D co-geometry 2**

**1.** Given the equation of a straight line is given as $2\left(x+2\right)=2\left(y-3\right)=z+1$.

 **(a)** Determine the vector equation of the straight line.

 **(b)** Hence, find the coordinates of a point Q that lies on the straight line such that $\left|OQ\right|=3\sqrt{2}$.

 **(a)** $2\left(x+2\right)=2\left(y-3\right)=z+1⟹\frac{x+2}{1}=\frac{y-3}{1}=\frac{z+1}{2}$

 The vector equation is $r=-2i+3j-k+t\left(i+j+2k\right)$.

 **(b)** Let $Q =\left(-2+t,3+t,-1+2t\right)$ be a point on the straight line.

 $\left|OQ\right|=3\sqrt{2}⟹OQ^{2}=18⟹\left(-2+t\right)^{2}+\left(3+t\right)^{2}+\left(-1+2t\right)^{2}=18⟹6t^{2}-2t-4=0$

 $⟹3t^{2}-t-2=0⟹\left(3t+2\right)\left(t-1\right)=0⟹t=-\frac{3}{2},1$

 $Q =\left(-\frac{7}{2},\frac{3}{2},-4\right)$ or $\left(-1,4,1\right)$.

**2.** Find the Cartesian equation of the plane passing through the points $A\left(4,-1,3\right)$ and $B\left(5,1,2\right)$ and containing the line : $r=4i-j+3k+λ\left(3i-j+k\right)$.

 $AB=i+2j-k$

Direction of the given line is $v=3i-j+k$

Normal of the required plane is $N=AB×v=\left|\begin{matrix}i&j&k\\1&2&-1\\3&-1&1\end{matrix}\right|=i+2j-7k$

Together with point $A\left(4,-1,3\right)$, the equation of the required plane is

 $1\left(x-4\right)+2\left(y+1\right)-7\left(z-3\right)=0$ or $x+2y-7z+19=0$

**3.** The position vectors of the points P, Q and R are $i+3k,2i+2j-k,i-j+k$respectively.

 Let the plane π contains the points P, Q and R.

 **(a)** Find a vector which is perpendicular to plane π.

 **(b)** Find the area of $∆PQR$.

 **(c)** Obtain the Cartesian equation of plane π.

 **(a)** $\rightharpoonaccent{PQ}=i+2j-4k, \rightharpoonaccent{PR}=-j-2k$

 $n=\rightharpoonaccent{PQ}×\rightharpoonaccent{PR}=\left|\begin{matrix}i&j&k\\1&2&-4\\0&-1&-2\end{matrix}\right|=-8i+2j-k$, which is a vector perpendicular to plane π.

 **(b)** Area of $∆PQR$ = $\frac{1}{2}\left|\rightharpoonaccent{PQ}×\rightharpoonaccent{PR}\right|=\frac{1}{2}\left|-8i+2j-k\right|=\frac{\sqrt{69}}{2}$

 **(c)** The normal of the plane is $-8i+2j-k$and$P\left(1,0,3\right)$ is on the plane.

 $π: -8\left(x-1\right)+2\left(y-0\right)-\left(z-3\right)=0$

 $π: 8x-2y+z=11$

**4.** Find the vector $n\_{1}$ normal to the plane $π\_{1}:r=\left(5i+j\right)+α\left(-4i+j+3k\right)+β\left(j+2k\right)$.

Write down a vector $n\_{2}$ normal to the plane $π\_{2}:3x+y-z=3$. Show that $4i+13j+25k$is normal to both $n\_{1}$ and $n\_{2}$. Given that the point $\left(1,1,1\right)$ lies on both $π\_{1}$ and $π\_{2}$.

 Write down the equation of line of intersection of $π\_{1}$ and $π\_{2}$ in the form of $r=a+tb$where t is a parameter.

 The two directional vectors $d=-4i+j+3k, e=j+2k$are on the plane, hence the normal is

$n\_{1}=d×e=\left|\begin{matrix}i&j&k\\-4&1&3\\0&1&2\end{matrix}\right|=-i+8j-4k$

$n\_{2}=3i+j-k$

 $n\_{2}×n\_{1}=\left|\begin{matrix}i&j&k\\3&1&-1\\-1&8&-4\end{matrix}\right|=4i+13j+25k$ , thus $4i+13j+25k$is normal to both $n\_{1}$ and $n\_{2}$.

 $n\_{2}×n\_{1}=4i+13j+25k$is a line parallel to the line of intersection of $π\_{1}$ and $π\_{2}$. Since $\left(1,1,1\right)$ lies on both $π\_{1}$ and $π\_{2}$, it must line on the intersection line.

 Therefore, the equation of intersection line is $r=\left(i+j+k\right)+t\left(4i+13j+25k\right)$ .

**5.** $OA=i+j-2k, OB=2i-j+k, OC=3i+j, OD=2i+3j-3k$.

 Point P divides the line AC in the ratio 2 : 1 internally.

 **(a)** Show that ABCD is a parallelogram.

 **(b)** Calculate the exact area of the parallelogram ABCD.

 **(c)** Find the position vector P and the angle APB in degrees corrected to one decimal place.

 **(a)** $AB=\left(2i-j+k\right)-\left(i+j-2k\right)=i-2j+3k$

$DC=\left(3i+j\right)-\left(2i+3j-3k\right)=i-2j+3k$

Since $AB=DC$, the opposite sides are equal and parallel, therefore ABCD is a parallelogram.

 **(b)** $AC=\left(3i+j\right)-\left(i+j-2k\right)=2i+2k$

Area of the parallelogram ABCD

 = $\left|AB×AC\right|=\left|\begin{matrix}i&j&k\\1&-2&3\\2&0&3\end{matrix}\right|=\left|-6i+3j+4k\right|=\sqrt{\left(-6\right)^{2}+3^{2}+4^{2}}=\sqrt{61}$.

 **(c)** $OP=\frac{\left(3i+j\right)+2\left(i+j-2k\right)}{3}=\frac{1}{3}\left(5i+3j-4k\right)$

 $PA=\left(i+j-2k\right)-\frac{1}{3}\left(5i+3j-4k\right)=\frac{1}{3}\left(-2i-2k\right)$

 $PB=\left(2i-j+k\right)-\frac{1}{3}\left(5i+3j-4k\right)=\frac{1}{3}\left(i-6j+7k\right)$

 $PA∙PB=\left|PA\right|\left|PB\right|\cos(APB⟹)-\frac{16}{9}=\left(\frac{1}{3}\sqrt{8}\right)\left(\frac{1}{3}\sqrt{86}\right)\cos(APB)⟹\cos(APB)=-\frac{16}{\sqrt{8}\sqrt{86}}$

$$angle APB=127.6°$$

**6.** Show that the lines with equations

 $r=7i-3j+3k+λ\left(3i-2j+k\right)$ and $r=7i-2j+4k+μ\left(-2i+j-k\right)$ intersect,

 and find the position vector of their point of intersection.

 The lines intersect if $7+3λ=7-2μ⇒3λ=-2μ …(1)$

 $-3-2λ=-2+μ⇒μ+2λ=-1 …(2)$

 $3+λ=4-μ⇒μ+λ=1 …(3)$

 $\left(2\right)-\left(3\right), λ=-2 …(4)$

 $\left(4\right)\downright \left(3\right), μ=3 …(5)$

 Since (4), (5) satisfy (1), equations (1) – (3) has a unique solution : $λ=-2, μ=3$.

 Therefore, the two given lines intersect.

 The point of intersection is $7i-3j+3k-2\left(3i-2j+k\right)=i+j+k$or in Cartesian form (1, 1, 1).

**7.** Given a sphere $x^{2}+y^{2}+z^{2}=126$

 **(i)**  Find the equations of the tangent planes to the sphere when x = 1 and y = 10.

 **(ii)** Find a point on the sphere that is farthest to the point (1, 2, 3). Hence, determine its distance.

 **(i) Method 1**

When x = 1 and y = 10, $1^{2}+10^{2}+z^{2}=126$ , $z=-5 or z=5$.

 The points on the sphere are $A\left(1,10,-5\right), B\left(1,10,5\right)$

 For point $A\left(1,10,-5\right)$ , the normal of the plane is $N\left(1,10,-5\right)$

 Hence, the equation of the plane is $1\left(x-1\right)+10\left(y-10\right)-5\left(z+5\right)=0$

$$x+10 y-5 z=126$$

 For point $B\left(1,10,5\right)$ , the normal of the plane is $N\left(1,10,5\right)$

 Hence, the equation of the plane is $1\left(x-1\right)+10\left(y-10\right)+5\left(z-5\right)=0$

$$x+10 y+5 z=126$$

 **Method 2 This method can be used for any surface other than sphere**

 Let $f\left(x,y\right)=z=\pm \sqrt{126-x^{2}-y^{2}}$

 The equation of the tangent plane to the surface $z=f\left(x,y\right)$ at $\left(x\_{0},y\_{0},z\_{0}\right)$ is given by

 $z-z\_{0}=f\_{x}\left(x\_{0},y\_{0}\right) \left(x-x\_{0}\right)+f\_{y}\left(x\_{0},y\_{0}\right) \left(y-y\_{0}\right)$

 Since $f\_{x}=\mp \frac{x}{\sqrt{126-x^{2}-y^{2}}} , f\_{y}=\mp \frac{y}{\sqrt{126-x^{2}-y^{2}}}$

 The equation of the tangent plane at $A\left(1,10,-5\right)$ is

 $z+5=\frac{1}{\sqrt{126-1^{2}-10^{2}}} \left(x-1\right)+\frac{10}{\sqrt{126-1^{2}-10^{2}}} \left(y-10\right)$ or $x+10 y-5 z=126$

 The equation of the tangent plane at $A\left(1,10,5\right)$ is

 $z-5=-\frac{1}{\sqrt{126-1^{2}-10^{2}}} \left(x-1\right)-\frac{10}{\sqrt{126-1^{2}-10^{2}}} \left(y-10\right)$ or $x+10 y+5 z=126$

 **(ii)**  The line joining the point (1, 2, 3) and the origin is

 $x=1+t, y=2+2t, z=3+3t$

 This line cuts the sphere, hence $\left(1+t\right)^{2}+\left(2+2t\right)^{2}+\left(3+3t\right)^{2}=126$

$$ t^{2}+2 t+1=9, t^{2}+2 t-8=0$$

$$t=-4 or t=2$$

 When $t=-4$, $x=-3, y= -6, z=-9$.

 Hence (-3, -6, -9) is a point on the sphere that is farthest to the point (1, 2, 3).

 Its distance is $\sqrt{\left(1+3\right)^{2}+\left(2+6\right)^{2}+\left(3+9\right)^{2}}=4 \sqrt{14}$

Note that this distance is also the sum of the distances from (1, 2, 3) to origin and the radius of the sphere =$\sqrt{1+4+9}+\sqrt{126}=4 \sqrt{14}$

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